Geometric and algebraic transience for block-structured Markov chains

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Outline

$M/G/1$-type Markov chains

Consider a time homogeneous discrete-time $M/G/1$-type Markov chain Φ_n with substochastic transition matrix:

$$P_M = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \dots \\ C_0 & A_1 & A_2 & A_3 & \dots \\ 0 & A_0 & A_1 & A_2 & \dots \\ 0 & 0 & A_0 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where C_0, A_i and B_i are square matrices of order $m < \infty$.

Its state space is $E = \bigcup_{i=0}^{\infty} L_i$, where $L_i = \{(i,j), 1 \leq j \leq m\}$.

$GI/M/1$-type Markov chains

A discrete-time $GI/M/1$-type Markov chain is of the following substochastic transition matrix:

$$P_{GI} = \begin{pmatrix} D_0 & \tilde{A}_0 & 0 & 0 & \dots \\ D_1 & A_1 & A_0 & 0 & \dots \\ D_2 & A_2 & A_1 & A_0 & \dots \\ D_3 & A_3 & A_2 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\tilde{A}_0$, A_i and D_i, $i \geq 0$ are square matrices of order m.

An illustrating example

Block-structured Markov chains are also called matrix-analytic models, which model many queueing problems.

Example: consider an $M/M/1$ queue in a Markovian environment, which is a continuous-time Markov chain $\{\Phi_t = (N(t), E(t)), t \geq 0\}$:

- $N(t)$ is the queue length at time t.
- $E(t)$ is a m-state CTMC with rates s_{ij}, $1 \leq i, j \leq m$.
- $N(t)$ is controlled by $E(t)$: when $E(t) = j$, the arrival rate is λ_j and the service rate is μ_j, provided that the server is busy at time t.

Let $m = 3$, then the generator Q of Φ_t is a QBD matrix

$$Q = \begin{pmatrix} B & A_2 & 0 & 0 & \dots \\ A_0 & A_1 & A_2 & 0 & \dots \\ 0 & A_0 & A_1 & A_2 & \dots \\ 0 & 0 & A_0 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $B = \begin{pmatrix} * & s_{12} & 0 \\ 0 & * & s_{23} \\ s_{31} & s_{32} & * \end{pmatrix}$, $A_0 = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$,

$A_1 = \begin{pmatrix} \bullet & s_{12} & 0 \\ 0 & \bullet & s_{23} \\ s_{31} & s_{32} & \bullet \end{pmatrix}$, $A_2 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$.

Some literature

- ▷ $M/G/1$-type Markov chains and $GI/M/1$-type Markov chains are typical block-structured Markov chains.
- ▷ See Neuts (1981, 1988) for introduction.
- ▷ See Latouche & Taylor (2003) for the criteria for transience and recurrence by drift conditions.
- ▷ See Hou & Liu (2004), Liu & Hou (2006) and Mao et al. (2012) for ergodicity.
- ▷ See Kijima (1993) and Li & Zhao (2002, 2003) for transience and subinvariant measures.
- ▷ Mao and Song (2014) investigated geometric and algebraic transience for DTMCs on a general state space.

Motivation

▷ Ramaswami (1990) revealed the duality relationship between the matrix $G(s)$ of $M/G/1$-type Markov chain and the matrix $R(s)$ of $GI/M/1$-type Markov chain as follows:

$$R(s) = \Delta^{-1} G^T(s) \Delta,$$

where Δ be the diagonal matrix with $\boldsymbol{\mu}^T$ on the diagonal.

▷ Zhao et al. (1999) extended Ramaswami's duality to derive: for stochastic transition matrices

$$P_M \text{ is positive recurrent iff } P_{GI} \text{ is transient.}$$

$$P_M \text{ is transient iff } P_{GI} \text{ is positive recurrent.}$$

▷ Based on the observation, it is natural to ask if

geometric and algebraic ergodicity of P_M (P_{GI}) correspond to geometric and algebraic transience of P_{GI} (P_M) with some additional conditions, respectively.

▷ We are motivated to answer the above question. Moreover, we will give a full characterization of geometric and algebraic transience for $M/G/1$-type or $GI/M/1$-type Markov chains.

▷ To investigate the quasi-stationary behavior, see Bean et al (1997) for QBD processes.

Outline

Definition 1: Let Φ_n be an irreducible DTMC on a countable state space. Then

(i) Φ_n is said to be transient if $\sum_{n=0}^{\infty} P_{ii}^{(n)} < \infty$;

(ii) Φ_n is said to be geometrically transient (GT) if there exists a constant $s > 1$ such that $\sum_{n=1}^{\infty} s^n P_{ii}^{(n)} < \infty$;

(iii) Φ_n is said to be ℓ-transient if there exists a positive integer ℓ such that $\sum_{n=1}^{\infty} n^{\ell} P_{ii}^{(n)} < \infty$.

Note: GT$\Rightarrow$ ℓ-transience for any $\ell \geq 1$ $\Rightarrow$ transience.

Define the first passage time on a non-empty subset $A \subset \mathbb{E}$ by

$$\tau_A = \inf\{n \geq 1 : \Phi_n \in A\}$$

and define the probability of Φ_n ever returning to A by

$$F_{iA} = P\{\tau_A < \infty | \Phi_0 = i\}.$$

When $A = \{i\}$, write simply $\tau_A = \tau_i$ and $F_{iA} = F_{ii}$.

Proposition 1: Suppose that the chain is irreducible. For $r(n) = n^\ell, \ell \in \mathbb{Z}_+$ or $r(n) = s^n, s \geq 1$, the following statements are equivalent.

(i) For some (then for all) $i \in \mathbb{E}$, $\sum_{n=0}^{\infty} r(n) P_{ii}^{(n)} < \infty$.

(ii) For some (then for all) $i \in \mathbb{E}$, $F_{ii} < 1$ and $E_i[r(\tau_i) 1_{\{\tau_i < \infty\}}] < \infty$.

(iii) For some (then for all) finite non-empty set $A \subset \mathbb{E}$, $\max_{i \in A} \sum_{n=0}^{\infty} r(n) P_{iA}^{(n)} < \infty$.

(iv) For some (then for all) finite non-empty set $A \subset \mathbb{E}$, $\max_{i \in A} E_i[r(\tau_A) 1_{\{\tau_A < \infty\}}] < \infty$ and $F_{jA} < 1$ for some $j \in A$.

Note: use the arguments in Chen (2004) to show (iii)$\Rightarrow$ (iv).

Outline

Recall M/G/1 and GI/M/1 chains

$$P_M = \begin{pmatrix} B_0 & B_1 & B_2 & B_3 & \dots \\ C_0 & A_1 & A_2 & A_3 & \dots \\ 0 & A_0 & A_1 & A_2 & \dots \\ 0 & 0 & A_0 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$P_{GI} = \begin{pmatrix} D_0 & \tilde{A}_0 & 0 & 0 & \dots \\ D_1 & A_1 & A_0 & 0 & \dots \\ D_2 & A_2 & A_1 & A_0 & \dots \\ D_3 & A_3 & A_2 & A_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Let

$$f_{(i,j),(i',j')}(n) = P\{\tau_{L_{i'}} = n, \Phi_n = (i',j')|\Phi_0 = (i,j)\}$$

be the probability that starting in the state (i,j) at time 0, the chain Φ_n first returns to level i' by hitting the phase j', after exactly $n \in \mathbb{N}_+$ transitions.

- In matrix form, we write $f_{ii'}(n) = (f_{(i,j),(i',j')}(n))$.
- Define the generating function $F_{ii'}(s) = \sum_{n=1}^{\infty} f_{ii'}(n)s^n$.
- Let $G(s) = F_{i+1,i}(s), i \geq 1$, which is independent of i because of the level-independent property of P_M.

Mean drift

- Throughout assume that $A := \sum_{k=0}^{\infty} A_k$ is irreducible.
- If A is stochastic, there is a unique invariant probability vector of A, denoted by $\boldsymbol{\mu}^T$, such that $\boldsymbol{\mu}^T A = \boldsymbol{\mu}^T$ and $\boldsymbol{\mu}^T e = 1$.
- Define $d = \boldsymbol{\mu}^T \boldsymbol{\nu} - 1$. Then

$$d = \boldsymbol{\mu}^T \left[\sum_{k=1}^{\infty} (k-1) A_k - A_0 \right] \boldsymbol{e}$$

is *mean drift* of the chains, which is a key quantity for analyzing stability and transience.

M/G/1-type Markov chains

Theorem 1: Let P_M be an irreducible M/G/1-type MC.

Case 1: both P_M and A are stochastic. If P_M is transient (i.e. $d > 0$), then P_M is GT.

Case 2: P_M is not stochastic but A is stochastic.

(i) If $d > 0$, then P_M is GT.

(ii) If $d = 0$ and G is irreducible, then P_M is transient but not ℓ-transient for any $\ell \geq 1$.

(iii) If $d < 0$, then P_M is GT iff $\min\{\phi_A, \phi_B\} > 1$ (ϕ_A is the radius of convergence of $A(z)$); and P_M is ℓ-transient for some $\ell \geq 1$ iff $\sum_{k=0}^{\infty} k^\ell A_k < \infty$ and $\sum_{k=0}^{\infty} k^\ell B_k < \infty$.

Case 3: A is not stochastic. P_M is GT.

Remarks about proof

(i) Using Proposition 1: through three basic equations

$$F_{00}(s) = sB_0 + \sum_{v=1}^{\infty} sB_v G^{v-1}(s) F_{10}(s).$$

$$F_{10}(s) = sC_0 + \sum_{v=1}^{\infty} sA_v G^{v-1}(s) F_{10}(s).$$

$$G(s) = \sum_{v=0}^{\infty} sA_v G^{v}(s).$$

(ii) Spectral properties + matrix analytical arguments+反证法

GI/M/1-type Markov chains

Theorem 2: Let P_{GI} be an irreducible GI/M/1-type MC.

Case 1: both P_{GI} and A are stochastic. If P_{GI} is transient ($d < 0$), then P_{GI} is GT iff $\phi_A > 1$, and P_{GI} is ℓ-transient for some $\ell \geq 1$ iff $\sum_{k=1}^{\infty} k^{\ell+1} A_k < \infty$.

Case 2: P_{GI} is not stochastic, but A is stochastic.

(i) If $d < 0$, then P_{GI} is GT iff $\phi_A > 1$; and P_{GI} is ℓ-transient for some $\ell \geq 1$ iff $\sum_{k=1}^{\infty} k^{\ell+1} A_k < \infty$.

(ii) If $d = 0$ and G is irreducible, then P_{GI} is not ℓ-transient for any $\ell \geq 1$.

(iii) If $d > 0$, then P_{GI} is GT.

Case 3: neither P_{GI} nor A is stochastic. P_{GI} is GT.

Remarks about proof

Using Proposition 1: we do not have similar equations like that for P_M, which causes difference.

For example, to consider algebraic transience for P_{GI}, define

$$F(s,z) = \sum_{i=1}^{\infty} F_{i0}(s) z^i, \quad D(z) = \sum_{k=1}^{\infty} D_k z^k, \quad s < 1, z < 1.$$

Then we can express $F_{10}(s)$ through (Hou and Liu 2004)

$$(zI - sA(z))F(s,z) = sz[D(z) - A_0 F_{10}(s)].$$

Extension to CTMCs

▷ Consider a CTMC Φ_t with irreducible and bounded generator Q and transition function $P_{ij}(t)$.

▷ Let $h > \bar{q}$ and define the h-uniformized chain $\Phi^h(n)$ with transition matrix $\hat{P}_{ij} = (I + h^{-1}Q)_{ij}, \quad i,j \in \mathbb{E}$. Using

$$P_{ij}(t) = (e^{tQ})_{ij} = \sum_{n=0}^{\infty} \hat{P}_{ij}(n) e^{-th} \frac{(th)^n}{n!},$$

shows that algebraic transience and geometric transience are equivalent for Φ_t and $\Phi^h(n)$.

▷ Using Theorems 1 and 2, we can get the classification of transience for continuous-time chains.

Outline

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